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ON THE TORSION OF A CURVE

BY PAUL SAUREL

THE object of this note is to call attention to a new way of stating the definition of the torsion of a curve, and to show that the new definition yields immediately the familiar formulas for the torsion of a space curve, of a geodesic curve, and of any curve traced on a surface.

The torsion of a curve at a given point is, by definition, the limit of the ratio of the angle between the osculating planes at the given point and at a neighboring point, to the arc joining these points. And in order that the sign of the torsion as thus determined shall agree with the sign as determined by Frenet's formulas it is necessary that the angle between the two planes be considered positive or negative according as the angle between the binormal at the given point and the principal normal at the neighboring point is obtuse or acute.*

This definition can be transformed into the following: the torsion of a curve at a given point is the limit of the ratio of the angle between the osculating plane at the given point and the principal normal at the neighboring point, to the arc joining the two points. To justify this definition we observe that the angle between the osculating plane at the given point and the tangent line at the neighboring point is an infinitesimal of the second order when the arc joining the two points is of the first order.† This tangent may therefore be said to lie in the osculating plane at the given point and it may thus be considered as the intersection of the two osculating planes. The angle between the two osculating planes may accordingly be replaced by the angle between the osculating plane at the given point and the principal normal at the neighboring point.

In applying this definition we can, of course, replace the angle between the osculating plane at the given point and the principal normal at the neighboring point by its sine, and this, in turn, by the cosine of the angle between the binormal at the given point and the principal normal at the neighboring

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*Cf. Kommerell und Kommerell, *Allgemeine Theorie der Raumkurven und Flächen*, vol. 1, p. 24.

†Cf. C. Jordan, *Cours d'analyse*, vol. 1, p. 465.

point. Moreover, if we remember the convention stated above concerning the sign of the angle between the two osculating planes, it is easy to see that this angle, with its proper sign, is equal to the negative of the cosine of the angle between the binormal at the first point and the principal normal at the second point.

It is worth noticing that our definition holds if the principal normal instead of being drawn as usual toward the centre of curvature be drawn in the opposite direction, provided the tangent, the principal normal and the binormal form a set of axes congruent with the axes of reference.

These results can of course be established by analysis. If we denote the arc of the curve by s, the radius of curvature by r, the radius of torsion by ρ, and the direction cosines of the tangent, the principal normal and the binormal by α, β, γ, l, m, n, λ, μ, ν, respectively, and if we suppose, moreover, that the positive direction of the principal normal passes through the centre of curvature and that the tangent, the principal normal, and the binormal form a system of axes congruent with the axes of reference, Frenet's formulas will take the form

$$\frac{d\alpha}{ds} = \frac{l}{r}, \qquad \frac{d\beta}{ds} = \frac{m}{r}, \qquad \frac{d\gamma}{ds} = \frac{n}{r}, \tag{1}$$

$$\frac{dl}{ds} = -\frac{\alpha}{r} - \frac{\lambda}{\rho}, \qquad \frac{dm}{ds} = -\frac{\beta}{r} - \frac{\mu}{\rho}, \qquad \frac{dn}{ds} = -\frac{\gamma}{r} - \frac{\nu}{\rho}, \tag{2}$$

$$\frac{d\lambda}{ds} = \frac{l}{\rho}, \qquad \frac{d\mu}{ds} = \frac{m}{\rho}, \qquad \frac{d\nu}{ds} = \frac{n}{\rho}. \tag{3}$$

The direction cosines of the binormal at any point being λ, μ, ν and those of the principal normal at an adjacent point being $l + \Delta l$, $m + \Delta m$, $n + \Delta n$, the cosine of the angle between the two lines is equal to the expression

$$\lambda(l + \Delta l) + \mu(m + \Delta m) + \nu(n + \Delta n), \tag{4}$$

or, more simply, to

$$\lambda\,\Delta l + \mu\,\Delta m + \nu\,\Delta n. \tag{5}$$

If in this expression we replace Δl, Δm, Δn by their values obtained from equations 2, we get

$$-\lambda\left(\frac{\alpha}{r} + \frac{\lambda}{\rho}\right)\Delta s - \mu\left(\frac{\beta}{r} + \frac{\lambda}{\rho}\right)\Delta s - \nu\left(\frac{\gamma}{r} + \frac{\nu}{\rho}\right)\Delta s + \cdots, \tag{6}$$

the omitted terms being of the second degree in Δs, and this in turn reduces to

$$-\frac{1}{\rho}\Delta s + \cdots. \tag{7}$$

Thus the limit of the fraction obtained by dividing the negative of this expression by the increment of the arc is the torsion.

If we suppose that the positive direction of the principal normal does not pass through the centre of curvature but that nevertheless the tangent, the principal normal and the binormal form a set of axes congruent with the axes of reference, we must replace l, m, n, λ, μ, ν in equations 1, 2, 3 by $-l$, $-m$, $-n$, $-\lambda$, $-\mu$, $-\nu$. The new formulas lead again to expression 7.

Let us apply our definition to any curve in space. We shall suppose that the coordinates x, y, z of any point on the curve are given as functions of the arc s measured from some given point of the curve, and we shall use primes to indicate differentiations with respect to s. Then, as is well known, the direction cosines of the tangent at any point are equal to x', y', z', while those of the principal normal and of the binormal are respectively proportional to

$$x'', \qquad y'', \qquad z'', \tag{8}$$

$$y'z'' - y''z', \quad z'x'' - z''x', \quad x'y'' - x''y'. \tag{9}$$

In both cases the factor of proportionality is the reciprocal of $\sqrt{x''^2 + y''^2 + z''^2}$.

From 8 it follows that the direction cosines of the principal normal at a neighboring point are proportional to

$$x'' + x'''\Delta s, \quad y'' + y'''\Delta s, \quad z'' + z'''\Delta s, \tag{10}$$

in which Δs denotes the increment of s. The factor of proportionality is now approximately equal to the reciprocal of $\sqrt{x''^2 + y''^2 + z''^2}$. From 9 and 10 we can calculate the cosine of the angle between the binormal at the given point and the principal normal at the adjacent point. If we divide this by $-\Delta s$ and take the limit we obtain at once the well known formula for the torsion $1/\rho$ of a space curve:

$$\frac{1}{\rho} = -\frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{x''^2 + y''^2 + z''^2}. \tag{11}$$

Let us next consider a geodesic curve on a given surface. Take the given point on the curve as the origin, the tangents to the lines of curvature through the point as axes of x and y, and the normal to the surface as axis of z. Denote, as usual, the first derivatives of z with respect to x and y by p and q, and the second derivatives by r, s, t. Then, because of our choice of axes, at the origin p, q and s are equal to zero, and at a neighboring point p and q are approximately equal to $rx'\Delta s$, $ty'\Delta s$ in which Δs denotes the arc between the two points and the primes indicate differentiation with respect to s.

The direction cosines of the tangent to the geodesic curve at the origin are

$$x',\ y',\ 0. \tag{12}$$

Since the principal normal at each point of a geodesic curve is, by definition, normal to the surface, its direction cosines are proportional to $-p$, $-q$, 1. At the origin the direction cosines of the principal normal to the geodesic line are therefore equal to

$$0,\ 0,\ 1, \tag{13}$$

and at an adjacent point of the geodesic line the direction cosines of the principal normal are approximately equal to

$$-rx'\Delta s,\ -ty'\Delta s,\ 1. \tag{14}$$

From 12 and 13 we find the direction cosines of the binormal to the geodesic line at the origin to be

$$y',\ -x',\ 0. \tag{15}$$

From 14 and 15 we can calculate the cosine of the angle between the binormal to the curve at the origin and the principal normal to the curve at an adjacent point. If we divide this by $-\Delta s$ we get the formula for the torsion $1/\rho_g$ of a geodesic curve

$$\frac{1}{\rho_g} = (r - t)x'y'. \tag{16}$$

If we denote the principal radii of curvature by R_1, R_2 and the angle between the geodesic line and one of the lines of curvature by θ, the equation assumes the more familiar form

$$\frac{1}{\rho_g} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\cos\theta\sin\theta = \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\sin 2\theta. \tag{17}$$

Finally, let us consider any curve drawn on a surface. At the given point P of the curve draw the tangent geodesic line. The two curves will have in common not only the point P but also an adjacent point Q. Let us denote by ω the angle that the principal normal to the given curve makes with the normal to the surface. Then, if we remember that the normal to the surface is also the principal normal of the geodesic line, it is not hard to see that the angle between the osculating plane to the given curve at P and the principal normal to the given curve at Q is equal to the angle between the osculating plane to the geodesic curve at P and the principal normal to the geodesic curve at Q plus the increment of ω. If then we denote the torsion of the tangent geodesic curve by $1/\rho_g$, it follows that the torsion $1/\rho$ of the given curve is determined by the equation

$$\frac{1}{\rho} = \frac{1}{\rho_g} + \omega', \tag{18}$$

in which the prime indicates differentiation with respect to the arc s. The positive direction of ω is from the binormal to the principal normal of the given curve.

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